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THE STRUCTURE OF ENERGY CONSERVING LOW-ORDER MODELS IN GEOPHYSICAL FLUID DYNAMICS

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Following pioneering work by Lorenz and Obukhov, low-order models (LOMs) remain an important tool in geophysical fluid dynamics. LOMs are commonly derived by applying the Galerkin method to the version of the Navier-Stokes equations appropriate for the problem at hand. Unfortunately, along with a number of highly attractive features, the method *per se* does not provide criteria for selecting modes, nor a guarantee that a model based on a particular set of modes will behave anything like the original system. As a consequence, LOMs employed in problems of geophysical fluid dynamics often lack fundamental conservation properties of the original equation, which may result in undesirable behaviors in such models. This accounts for the interest in general principles for constructing effective models.

Obukhov (1969) insisted that a LOM should retain three characteristic features of the original system: quadratic nonlinearity; and in the absence of forcing and dissipation, conservation of energy and of phase space volume. Still, the class of models satisfying these requirements (Obukhov called them hydrodynamic type systems (HTSs)) is too broad. We introduce a significant subclass that consists of coupled 3-mode nonlinear systems

$$\begin{aligned} \dot{v}_1 &= p v_2 v_3 + b v_3 - c v_2, \\ \dot{v}_2 &= q v_3 v_1 + c v_1 - a v_3, \\ \dot{v}_3 &= r v_1 v_2 + a v_2 - b v_1; \quad p+q+r=0. \end{aligned} \tag{1}$$

known in mechanics as Volterra gyrostats. The reason behind this is that coupled gyrostats are HTSs and they arise as energy conserving LOMs in many important problems of geophysical fluid dynamics and turbulence (Gluhovsky and Agee 1997). Earlier, Obukhov (1973) suggested systems of coupled Euler gyroscopes (Eqs. (1) without linear terms) for this purpose. However, such linear terms arise in LOMs due to various factors peculiar to geophysical fluid dynamics (stratification, rotation, and topography). Unlike ordinary viscous terms, linear (gyrostatic) terms in Eqs. (1) do not affect the conservation of energy $\sum v_i^2/2$ or phase volume $\sum \partial \dot{v}_i / \partial v_i = 0$.

The equations for a gyrostat that has only two nonlinear terms and one pair of gyrostatic terms,

$$\begin{aligned} \dot{v}_1 &= -q v_2 v_3, \\ \dot{v}_2 &= q v_3 v_1 - a v_3, \\ \dot{v}_3 &= \phantom{q v_3 v_1 - {}} a v_2, \end{aligned} \tag{2}$$

with linear friction and a constant external force added, become the equations of the celebrated Lorenz (1963) model of convection (Gluhovsky 1982),

$$\dot{x} = \sigma(y-x), \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz.$$

System (2) has two independent quadratic invariants, the energy and the squared angular momentum, or their combination $I = v_3^2/2 + (a/q)v_1$ and the energy $E = (v_1^2 + v_2^2 + v_3^2)/2$. It is interesting to see the relation of these integrals to fundamental quantities in a convective flow (kinetic energy *K*, potential energy *U* and available potential energy *A*) where *K*+*U* (total mechanical energy) and *K*+*A* are conserved. We show that *I* and *E* are proportional to *K*+*U* and *K*+*U*+λ(*K*+*A*), respectively, where λ is a constant.

Another advantage of using coupled gyrostats as the basic structure for LOMs is that the failure for a LOM to be equivalent to a system of coupled gyrostats usually indicates a violation of fundamental conservation properties. One example is provided by the important Howard – Krishnamurti (1986) model that has also received particular attention in studies of Rayleigh-Bénard convection. Howard and Krishnamurti noticed that the model possessed trajectories going to infinity that they rightly attributed to deficiencies of the truncation. Indeed, Thiffeault and Horton (1996) found that the model lacks energy conservation in the dissipationless limit, which can be remedied by adding one term in the Galerkin expansion of the stream function. This operation adds terms to the original model which gives it a coupled gyrostat form

$$
\begin{array}{l|l|l|l|l|l}
\dot{v}_1 = & a_1 v_4 & & +p_3 v_2 v_3 & & -\gamma_1 v_1, \\
\dot{v}_2 = & & & q_3 v_3 v_1 & & -\gamma_2 v_2, \\
\dot{v}_3 = & & -a_2 v_6 & +r_3 v_1 v_2 & & -\gamma_3 v_3, \\
\dot{v}_4 = & q_1 v_5 v_1 - a_1 v_1 & & & -v_6 v_2 & -\gamma_4 v_4, \\
\dot{v}_5 = & -q_1 v_1 v_4 & & & & -\gamma_5 v_5 + f, \\
\dot{v}_6 = & & q_2 v_7 v_3 + a_2 v_3 & & +v_2 v_4 & -\gamma_6 v_6, \\
\dot{v}_7 = & & -q_2 v_3 v_6 & & & -\gamma_7 v_7 + 2f,
\end{array}
\qquad (3)
$$

and thereby ensures energy conservation and boundedness of trajectories. The integrals of system (3) $I = (v_1^2 + v_2^2 + v_3^2)/2 + (a_1/q_1)v_5 - (a_2/q_2)v_7$ and $E = \sum v_i^2/2$ have exactly the same interpretations as the integrals I and E in the Lorenz model above.

The Charney and DeVore (1979) model of a barotropic atmosphere over topography also results in a LOM in the form of coupled gyrostats. Below are the equations for the second approximation for this problem,

$$
\begin{array}{l|l||l|l|l}
\dot{v}_1 = & -q v_2 v_3 \quad -c v_2 & +P v_5 v_6 & & -\gamma_1 v_1, \\
\dot{v}_2 = & q v_3 v_1 \; -a v_3 \; +c v_1 & & -P v_4 v_6 & -\gamma_2 v_2, \\
\dot{v}_3 = & a v_2 & & & -\gamma_3 v_3 + f, \\
\dot{v}_4 = & -\tilde{q} v_5 v_6 \quad -\tilde{c} v_5 & & -Q v_6 v_2 & -\gamma_4 v_4, \\
\dot{v}_5 = & \tilde{q} v_6 v_4 - \tilde{a} v_6 + \tilde{c} v_4 & +Q v_6 v_1 & & -\gamma_5 v_5 \\
\dot{v}_6 = & \tilde{a} v_5 & +R v_1 v_5 & -R v_2 v_4 & -\gamma_6 v_6
\end{array}
$$

while the first approximation is simply the single gyrostat in the upper left corner. Linear gyrostatic terms with coefficients a and $\tilde{a}$ are caused by topography, while those with coefficients c and $\tilde{c}$ are caused by rotation. Thus, an increase in the order of approximation results in adding certain new gyrostats to the model while introducing further physical effects results in the appearance of additional gyrostatic terms in the model. This situation is typical for LOMs in geophysical fluid dynamics and suggests a modular approach to the construction of LOMs.

The following cascade system of coupled gyrostats (a version of one introduced by Gluhovsky (1989)) can be used in studies of turbulence:

$$
\begin{array}{lll}
\dot{v}_1 = & p_1 v_2 v_3 & -\lambda_1 v_1 + f, \\
\dot{v}_2 = & q_1 v_3 v_1 - a_1 v_3 \; + p_2 v_3 v_4 & -\lambda_2 v_2, \\
\dot{v}_3 = & r_1 v_1 v_2 + a_1 v_2 \; + q_2 v_4 v_2 - a_2 v_4 + \ldots & -\lambda_3 v_3, \\
\dot{v}_4 = & r_2 v_2 v_3 + a_2 v_3 + \ldots & -\lambda_4 v_4, \\
\ldots & \ldots & \ldots \\
\dot{v}_{n-2} = & \ldots + p_{n-2} v_{n-1} v_n & -\lambda_{n-2} v_{n-2}, \\
\dot{v}_{n-1} = & \ldots + q_{n-2} v_n v_{n-2} - a_{n-2} v_n & -\lambda_{n-1} v_{n-1}, \\
\dot{v}_n = & r_{n-2} v_{n-2} v_{n-1} + a_{n-2} v_{n-1} & -\lambda_n v_n.
\end{array}
\qquad (4)
$$

In the absence of forcing and dissipation ($f = 0$, $\lambda_i \equiv 0$), system (4) conserves energy and phase volume. Upon making certain natural assumptions about the coefficients (in line with the Kolmogorov hypothesis about the self-similarity of a cascade), $p_{i+1}/p_i = d$, $a_{i+1}/a_i = d^{2/3}$, $\lambda_{i+1}/\lambda_i = d^2$, the behavior of the system is determined by three dimensionless parameters: $R = |p_1| f/\lambda_1^2$ (an analog of the Reynolds number), $\sigma = a_1/\sqrt{|p_1| f} = (a_1/\lambda_1)/\sqrt{R}$ (taking into account the relative influence of gyrostatic terms), and $\delta = q_i/r_i$

At small R, the system has a stable stationary solution ("laminar flow") that evolves with R increasing from a form with a single nonzero component ($v_1 = f/\lambda_1$, $v_i = 0$, $i > 1$) to those with more than one nonzero component. Further increase in R brings the system into a chaotic ("turbulent") regime (Fig. 1). System (4) may be viewed as a severe truncation of the Navier-Stokes equations where only one mode $v_i(t)$ represents all Fourier modes in the octave shell with inner radius $1/\sqrt{d}\,k_i$ and outer radius $\sqrt{d}\,k_i$, and wave numbers k_i are logarithmically spaced $k_i = d^i k_0$ (a so-called shell model). Then time averages $\overline{v_i^2} = 2\int_{1/\sqrt{d}\,k_i}^{\sqrt{d}\,k_i} E(k)dk$ are the energies contained in such shells and $\overline{v_i^2} \propto d^{-2i/3}$ corresponds to the Kolmogorov – Obukhov law $E(k) \propto k^{-5/3}$. Fig. 2 demonstrates that system (4) in turbulent regimes possesses an inertial subrange (that broadens as R is increasing) in which its spectral behavior follows the Kolmogorov – Obukhov law.

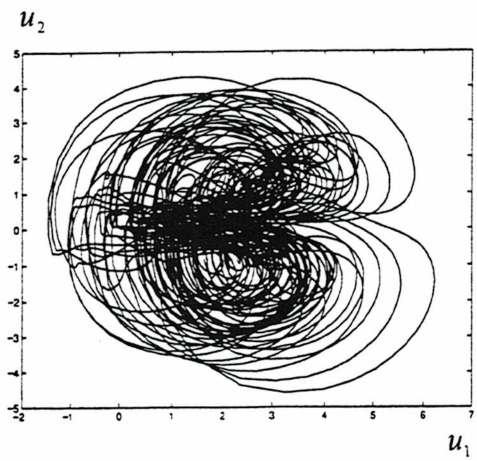

Fig. 1 Chaotic regime in system (4) at $\sigma = 0.2$, $\delta = 4.0$, $R = 35000$. Projection of the trajectory on the plane (u_1, u_2).

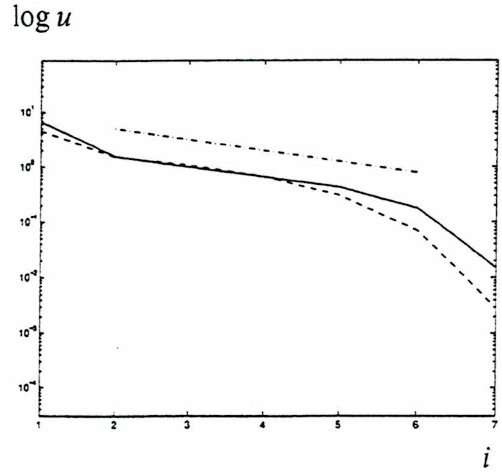

Fig. 2 Energy distributions in system (25) at $\sigma = 0.2, \delta = 4.0$ $R = 35000$ (solid curve) and $R = 17000$ (dashed curve); dash-dotted line corresponds to the Kolmogorov-Obukhov law.

We agree with Brown and Chua (1992) that "there is a pressing need for new nonlinear techniques that employ a building block approach whereby simple well-understood components are used to construct models of complex dynamical systems". We believe that coupled gyrostats could play the role of above building blocks in problems of geophysical fluid dynamics, including turbulence. They possess fundamental conservation properties of the original equations and all their trajectories are bounded. An increase in the order of approximation and/or adding new mechanisms result in adding new gyrostats to the system or gyrostatic terms to existing gyrostats Relationships are also established between integrals of motion in the fluid and those in gyrostats. Failure for a LOM to have a gyrostatic structure indicates that proper energy integrals are not conserved. Thus, coupled gyrostatic structure ensures that LOMs retain some significant physics of the original Navier-Stokes equations.

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